BILINEAR FORMS AND THE ADJOINT OF A LINEAR MAP

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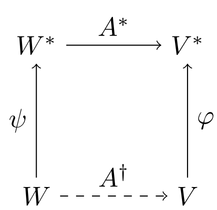
1. The Adjoint of a linear map

Let V be a real vector space and V^* its dual. Suppose we have a linear map $\varphi : V \to V^*$, then we can define a bilinear form on V by $B(x, y) = \varphi(x)(y)$. Conversely, given a bilinear form we can define a mapping from $V \to V^*$ by $\varphi(x) = B(\cdot, x)$. We say a bilinear form is non-degenerate if the associated φ is injective. For a finite dimensional vector space, if we have a non-degenerate bilinear form, then we have a canonical identification of V with its dual space $V \cong V^*$ via the associated map φ.

Now, given two vector spaces V and W and a linear map $A : V \to W$ between them, we have an associated map $A^* : W^* \to V^*$, called the dual of A, which pulls back functionals to X. That is, A^* is defined by $A^*(f) = f \circ A$ for $f \in W^*$. However, if we have non-degenerate bilinear forms on both V and W we get isomorphisms $\varphi : V \to V^*$ and $\psi : W \to W^*$. How does A^* behave under these identifications? We have the diagram below; define $A^\dagger = \varphi^{-1} \circ A^* \circ \psi$ so that it commutes. As we'll see, the operator $A^\dagger$ will be the adjoint (with respect to the bilinear forms).



Theorem 1.1. *Let B_V and B_W be non-degenerate bilinear forms on V and W respectively. Define $A^\dagger = \varphi^{-1} \circ A^* \circ \psi$, where φ and ψ are the isomorphisms associated to the bilinear forms. Then for $v \in V$ and $w \in W$*

$$B_W(Av, w) = B_V(v, A^\dagger w)$$

Proof. The definition of $A^\dagger$ can be rewritten as commutativity of the above diagram: $A^* \circ \psi = \varphi \circ A^\dagger$. Take $w \in W$,

$$\begin{aligned} \psi(w) \circ A = A^*(\psi(w)) &= (A^* \circ \psi)(w) \\ &= (\varphi \circ A^\dagger)(w) = \varphi(A^\dagger w). \end{aligned}$$

Now evaluate on $v \in V$:

$$\begin{aligned} B_W(Av, w) = \psi(w)(Av) &= (\psi(w) \circ A)(v) \\ &= \varphi(A^\dagger w)(v) = B_V(v, A^\dagger w), \end{aligned}$$

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as desired. ∎

Can we find the representation of $A^\dagger$ in coordinates? Indeed we can. Let's first fine the matrix representation of the map φ associated to a bilinear form B with respect to a basis $E_1, \dots, E_n$ for V and the dual basis $E_1^*, \dots, E_n^*$ for V^*.

Lemma 1.2. *The matrix representation of φ in the basis $E_1, \dots, E_n$ and the corresponding dual basis is $[\varphi] = (B(E_i, E_j))$.*

Proof. We can compute

$$\varphi(E_j) = \sum_{i=1}^{n} \varphi_{ij} E_i^*$$

and so $B(E_i, E_j) = \varphi(E_j)(E_i) = \varphi_{ij}$. ∎

This $[\varphi]$ is called the matrix representation of B and will will now denote it by $Q = (q_{ij}) = (B(E_i, E_j))$. An immediate consequence of this is that Q is an invertible matrix since φ is an isomorphism. The following lemma provides a useful expression for a bilinear form in coordinates.

Lemma 1.3. *If $E_1, \dots, E_n$ is a basis for V and $q_{ij} = B(E_i, E_j)$ then for $Q = (q_{ij})$ we have*

$$B(x, y) = [x] \cdot Q\, [y]\,,$$

where $[\cdot]$ is the coordinate representation with respect to the relevant basis.

Proof. This follows from computation: write $x = \sum_i x^i E_i$ and $y = \sum_j y^j E_j$, then

$$B(x, y) = B\left(\sum_{i=1}^{n} x^i E_i, \sum_{j=1}^{n} y^j E_j\right) = \sum_{i,j=1}^{n} x^i y^j B(E_i, E_j) = \sum_{i,j=1}^{n} x^i y^j q_{ij}.$$

However,

$$Q\,[y] = \sum_{j,k=1}^{n} y^j q_{kj}\,[E_k]$$

so that

$$[x] \cdot Q\,[y] = \sum_{k,j=1}^{n} x^i y^j q_{kj}\,[E_i] \cdot [E_k] = \sum_{i,j=1}^{n} x^i y^j q_{ij}$$

since the coordinate representations of the E_i are orthogonal.

∎

The preceeding arguments all work for W with its bilinear form. We can now compute the coordinate representation of the adjoint $A^\dagger$. First we do the simple case of $\mathbb{R}^n$ with the dot product. Here the adjoint is the transpose.

Lemma 1.4. *Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be linear, then in the standard basis we have $[A^\dagger] = [A]^t$.*

Proof. We note for a linear map L that $L_{ij} = e_i \cdot L e_j$. Therefore,

$$A^\dagger_{ij} = e_i \cdot A^\dagger e_j = A e_i \cdot e_j = e_j \cdot A e_i = A_{ji}.$$

∎

Now, for the general case.

Theorem 1.5. *In a basis $E_1, \ldots, E_n$ for V and $F_1, \ldots, F_m$ for W, the adjoint $A^\dagger$ has the representation*

$$[A^\dagger] = Q^{-1}[A]^t P,$$

where Q and P are the matrix representations of B_V and B_W, respectively.

Proof. We have from the previous lemmas that

$$\begin{aligned}[v] \cdot Q \left[A^\dagger\right][w] &= B_V(v, A^\dagger w) \\ &= B_W(Av, w) = [A]\,[v] \cdot P\,[w] = [v] \cdot [A]^t\, P\,[w]\,.\end{aligned}$$

This is true for all v and w, so $Q\left[A^\dagger\right] = [A]^t P$ and the result follows.

□

This was all under the assumption that our vector spaces were real. Adjoints on a complex vector space typically include conjugation; for example, a complex inner product is not bilinear but hermitian. Hence we don't usually have a canonical isomorphism between our vector spaces and their duals. More care is needed in this case.

2. Automorphism of bilinear forms

Let V and W be two vector spaces with non-degenerate bilinear forms B_V and B_W, respectively. What are morphisms between spaces like this? That is, what are the functions $f : V \to W$ that respect the bilinear forms?

Theorem 2.1. *Let $f : V \to W$ be a surjective function such that*

$$B_W(f(x), f(y)) = B_V(x, y),$$

then f is linear and an isomorphism.

Proof. We compute

$$\begin{aligned}&B_W(f(\alpha x + \beta y) - \alpha f(x) - \beta f(y), w) \\ &\qquad = B_W(f(\alpha x + \beta y), w) - \alpha B_W(f(x), w) - \beta B_W(f(y), w).\end{aligned}$$

Since f is surjective, there is a $z \in V$ such that $f(z) = w$. So our calculation becomes

$$\begin{aligned}B_W(f(\alpha x + \beta y) &- \alpha f(x) - \beta f(y), w) \\ &= B_W(f(\alpha x + \beta y), f(z)) - \alpha B_W(f(x), f(z)) - \beta B_W(f(y), f(z)) \\ &= B_V(\alpha x + \beta y, z) - \alpha B_V(x, z) - \beta B_V(y, z) \\ &= 0.\end{aligned}$$

Since this is zero for all w, by non-degeneracy we get

$$f(\alpha x + \beta y) - \alpha f(x) - \beta f(y) = 0.$$

So f is linear. It is already surjective so if we can show it's injective it will be an isomorphism. But this follows from similar reasoning: suppose $f(x) = 0$, then

$$B_V(x, z) = B_W(f(x), f(z)) = 0.$$

Since this is true for all z we get $x = 0$ by non-degeneracy.

□

The previous Theorem allows us to restrict our attention to linear isomomorphisms. We define the pullback of a bilinear form B similarly to how we defined the dual of a linear map. That is, if $A : V \to W$ is linear then we get a map $A^* : W^* \otimes W^* \to V^* \otimes V^*$ by $A^*B(x, y) = B(A(x), A(y))$.

Let's specialize to $V = W$ and $B_V = B_W = B$. Define $\mathrm{Aut}(B)$ to be

$$\mathrm{Aut}(B) = \{A : V \to V \mid A^*B = B\} = \{A : V \to V \mid B(A(\cdot), A(\cdot)) = B(\cdot, \cdot)\},$$

the set of all B-automorphisms of V. Since $Id \in \mathrm{Aut}(B)$ and $A \in \mathrm{Aut}(B)$ implies $A^{-1} \in \mathrm{Aut}(B)$ we get $\mathrm{Aut}(B)$ is a group under composition. So $\mathrm{Aut}(B) \leq \mathrm{Aut}(V)$.

If we take B to be the dot product on $\mathbb{R}^n$ then $\mathrm{Aut}(B) = O(n)$. If $B = \omega$ is a symplectic form then $\mathrm{Aut}(B) = Sp(2n, \mathbb{R})$, the symplectic group. If our bilinear form is

$$B(x, y) = x_1y_1 + \cdots + x_py_p - x_{p+1}y_{p+1} - \cdots - x_{p+q}y_{p+q}$$

on $\mathbb{R}^{p+q}$ then $\mathrm{Aut}(B) = O(p, q)$, and in the special case of $\mathbb{R}^{3,1}$ $\mathrm{Aut}(B) = O(3, 1)$ is the Lorentz group. The above holds for sesquilinear forms too; so, for example, if B is the complex inner product on $\mathbb{C}^n$ then we get $\mathrm{Aut}(B) = U(n)$.

How do elements of $\mathrm{Aut}(B)$ look in coordinates? The condition that $A^*B = B$ gives us

$$B(x, y) = B(Ax, Ay) = B(x, A^\dagger Ay) \quad \Longrightarrow \quad B(x, (Id - A^\dagger A)y) = 0$$

we get from non-degeneracy that if $A^*B = B$ then $A^\dagger A = Id$. Let's choose a basis $E_1, \ldots, E_n$ for V and define $Q = (B(E_i, E_j))$. So if A is a B-automorphism then we can compute

$$A^\dagger A = Id \quad \Longrightarrow \quad Q^{-1}A^tQA = Id.$$

Theorem 2.2. *An automorphism A is a B-automorphism if and only if $A^tQA = Q$.*

Proof. We have just seen necessity. For sufficiency we compute

$$B(Ax, Ay) = A\,[x] \cdot QA\,[y] = [x] \cdot A^tQA\,[y] = [x] \cdot Q\,[y] = B(x, y).$$

So $A^*B = B$ as desired.

□

As a quick example, if we have the dot product then our matrix $Q = Id$ so A is an orthogonal transformation if and only if $A^tA = Id$.